Functors, Fixed Points, and Algebras; Oh My!

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For a given endofunctor $F : \mathcal{C} \to \mathcal{C}$, let

- $(\mu F, in)$ denote its initial algebra $(in : F(\mu F) \to \mu F)$ A *catamorphism* h is the unique arrow from an initial algebra to any other a; it must be the solution to the equation $h \circ in = a \circ Fh$.
- $(\nu F, out)$ denote its terminal coalgebra $(out : \nu F \to F(\nu F))$ An *anamorphism* of a coalgebra c is the unique solution to $c \circ h = Fh \circ out$.

1 Special Fixed Points

1.1 Carrier of Initial Algebra Implies Fixed Point

Assume that $(\mu F, in)$ exists; then $(F(\mu F), F(in))$ is also an F-algebra (because $F(in) : F(F(\mu F)) \to F(\mu F)$) and the following diagram exists in $\mathcal{C}$:

$$\begin{tikzcd}
F(\mu F) \arrow[r, "F(\phi_F(F(in))"] \arrow[d, "in"'] \arrow[dr, phantom, "\circ"] & F(F(\mu F)) \arrow[r, "F(in)"] \arrow[d, "F(in)"'] \arrow[dr, phantom, "\circ"] & F(\mu F) \arrow[d, "in"] \\
\mu F \arrow[r, "\phi_F(F(in))"'] & F(\mu F) \arrow[r, "in"'] & \mu F
\end{tikzcd}$$

where $\phi_F(F(in))$ is the catamorphism of $F(in)$: the unique arrow guaranteed to exist by initiality of $(\mu F, in)$ such that the left square commutes. The right square trivially commutes and is rendered only for convenience.

We see that $in \circ \phi_F(F(in))$ (the bottom two arrows of the diagram) form an algebra homomorphism from $(\mu F, in)$ to itself. By initiality, (i.e., since the composite arrow forms an algebra homomorphism from $(\mu F, in)$ to itself, it must be that $\phi_F(in) = id$), this must be the identity: $in \circ \phi_f(F(in)) = 1_{\mu F}$

The top-left and middle edges compose to give $F(in) \circ F(\phi_F(F(in))$, which is just $F(in \circ \phi_F(F(in)))$ (because functors distribute over compostion), which we know to be $F(1_{\mu F})$, which is $1_{F(\mu F)}$ (because functors send identity arrows to identity arrows). By commutation of the left square, the left and bottom-left arrows' composition, $\phi_F(F(in)) \circ in = 1_{F(\mu F)}$.

All told, then, in is an isomorphism with inverse $in^{-1} = \phi_F(F(in))$. That is, μF satisfies the equation $\mu F \simeq F(\mu F)$, so μF is a fixed point of F. (This is apparently known as "Lambek's Lemma".)

1.2 Carrier of Final Coalgebra Implies Fixed Point

The above argument dualizes in a straightforward way.

$$\begin{tikzcd}
F(\nu F) \arrow[dr, phantom, "\circ"] & F(F(\nu F)) \arrow[l, "F(\psi_F(F(out))"'] \arrow[dr, phantom, "\circ"] & F(\nu F) \arrow[l, "F(out)"'] \\
\nu F \arrow[u, "out"] & F(\nu F) \arrow[l, "\psi_F(F(out))"] \arrow[u, "F(out)"] & \nu F \arrow[l, "out"] \arrow[u, "out"']
\end{tikzcd}$$

Where $\psi_F(F(out))$ is the anamorphism guaranteed by terminality of $(\nu F, out)$. $\psi_F(F(out)) \circ out = 1_{\nu F}$ by terminality (and $\psi_F(out) = id$; also note that dualization has swapped which half of the isomorphism follows immediately!). In the other direction, $out \circ \psi_F(F(out)) = F(\psi_F(F(out))) \circ F(out) = F(\psi_F(F(out)) \circ out) = F(1_{\nu_F}) = 1_{F(\nu_F)}$.

Thus out is an isomorphism with inverse $out^{-1} = \psi_F(F(out))$ and ν_F satisfies $\nu_F \simeq F(\nu F)$, making it another fixed point.

1.3 Other Fixed Points

We know that *any* fixed point of F in fact, call it θF, has the property of being isomorphic to $F(\theta F)$.

1.3.1 An Example or Two of Fixed Points

For the purpose of this section, consider the lovely binary tree functor on the category **Set**, which has (countable) (co)products, initial object $\emptyset$ and terminal object 1; i.e., $Tx = 1 + x \times x$. Then,

- One fixed point, in fact the smallest, of F is the collection of *finite* binary trees with 1 at its leaves. This is the usual thing obtained by inflation.
- The largest is binary trees with *countable* (including finite) paths (and 1 leaves). This cannot be defined by inflation but is clearly closed under T: taking the product of two such such objects is clearly another such object.
- Another intermediate structure θT is less obvious: trees which may descend *left* countably many times but *right* only finitely many times. Again, we cannot grow this by inflation but can argue that the product of any two such objects is, indeed, another such object.

Note that, indeed, as we might expect, $\mu T \subsetneq \theta T \subsetneq \nu T$. The "other θT" (which swaps left and right) is also between μT and νT, but is not comparable to θT: fixed points form a partial order with a single bottom and single top.

Consider a different functor, the diagonal product functor $\Delta x = x \times x$, which is like T except that it omits the "1+" part. In this case,

- $\mu\Delta = \emptyset$. There's nothing to force us away and $\Delta\emptyset = \emptyset \times \emptyset \simeq \emptyset$.
- The terminal object 1 is $\nu\Delta$: $\Delta 1 = 1 \times 1 \simeq 1$. For ease of understanding, this is the singleton set whose element represents a tree that is its own root's left and right child.
- There are no other fixed points of Δ. (Stated without proof!)

As before, $\mu\Delta \subseteq \nu\Delta$.

1.4 Induced Duals

Because of the two isomorphisms above, we know that $(\mu F, \phi_F(F(in))) = (\mu F, in^{-1})$ exists and is a coalgebra; similarly, $(\nu F, \psi_F(F(out))) = (\nu F, out^{-1})$ exists and is an algebra. That is, we have these diagrams (on the left are F-algebras and on the right are F-coalgebras; both diagrams take place in $\mathcal{C}$):

$$\begin{tikzcd}
F(\mu F) \arrow[r, "F(\phi_F(out^{-1}))"] \arrow[d, "in"'] \arrow[dr, phantom, "\circ"] & F(\nu F) \arrow[d, "out^{-1}"] \\
\mu F \arrow[r, "\phi_F(out^{-1})"'] & \nu F
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\mu F) \arrow[r, "F(\psi_F(in^{-1}))"] \arrow[dr, phantom, "\circ"] & F(\nu F) \\
\mu F \arrow[u, "in^{-1}"] \arrow[r, "\psi_F(in^{-1})"'] & \nu F \arrow[u, "out"']
\end{tikzcd}$$

where, again, $\phi_F(out^{-1}) = \phi_F(\psi_F(F(out)))$ is the unique arrow that makes the diagram commute, guaranteed to exist by initiality of $(\mu F, in)$ and, dually, $\psi_F(\phi_F(F(in))) = \psi_F(in^{-1})$ by terminality of $(\nu F, out)$. Note that this existence argument does *not* make these arrows equal.

1.4.1 Example Induced Coalegbra

For this section, we're going to write fuctions in Haskell, though only for its syntax. Please read all definitions as strict and total!

Consider the List (of As) **Set** endo-functor $ListF\ x = 1 + A \times x$. Let's give it a Haskell rendering, choosing $A = \text{Int}$ just to dodge polymorphism:

```haskell
data ListF rec = FNil | FCons Int rec
```

μL is (isomorphic to) the traditional LISP-y lists whose elements are of type A:

```haskell
data MuListF = MLNil | MLCons Int MuListF
```

$in : L(\mu L) \to \mu L$ sends 1 to the empty list and $a \times l$ to the list whose head is a and tail is l:

```
_in :: ListF MuListF -> MuListF
_in FNil = MLNil
_in (FCons i l) = MLCons i l
```

Moreover, we know (stated without proof) that L's catamorphism is an uncurried foldr: looking at the type, we see $\phi_L : \forall_X.(LX \to X) \to (\mu L \to X)$. (Expanding things a bit, this is $\forall_X.((1 + A \times X) \to X) \to \mu L \to X$ which is iso to $\forall_X.(A \to X \to X) \to X \to \mu L \to X$.) In particular, it is

```
phiL :: forall a . (ListF a -> a) -> MuListF -> a
phiL f MLNil = f (FNil)
phiL f (MLCons i l) = f (FCons i (phiL f l))
```

As required, $\phi_L(in) = id$:

```
phiL _in MLNil = _in FNil = MLNil

phiL _in (MLCons i l)
  = _in (FCons i (phiL f l))
  = _in (FCons i l)        -- induction
  = MLCons i l
```

$in^{-1} = \phi_L(L(in))$ instantiates ϕ_L with $X = L(\mu L)$. The function $L(in) : L(L\mu L) \to L\mu L$ then is just `fmap in`. So $in^{-1} = \phi_L(L(in)) : \mu L \to L\mu L$ is

```
phiL (fmap _in) MLNil = (fmap _in) FNil = FNil

phiL (fmap _in) (MLCons i l)
  = (fmap _in) (FCons i (phiL (fmap _in) l))
  = FCons i (_in (phiL (fmap _in) l))
  = FCons i l
```

1.4.2 Example Induced Algebra

νL is the set of possibly-infinite lists: it contains all of μL as well as the non-terminating lists. *out* sends nil to the left 1 and a list with cons cell top to the right product of its head and tail. So what is ψ_L? We know from looking at the diagram that it must have type $\forall_X.(X \to LX) \to X \to \nu L$.

$((X \to LX) \to \beta) = (X \to (1 + A \times X)) \to \beta$