Cheatsheet derived from Dummit & Foote.

1 Basics

A **group** (§1.1, p16) is an associative binary operation (denoted by concatenation), closed over a set G with a **identity** ($\exists!_{1\in G}.\forall_{g\in G} 1g = g1 = g$) and **inverses** ($\forall_{g\in G}.\exists!_{g^{-1}\in G}.gg^{-1} = g^{-1}g = 1$).

A group G is **abelian** if also $\forall_{a,b\in G}.ab = ba$.

The **order** (p20) of an element $g \in G$, $|g|$, is the smallest $n \in \mathbb{N}^+$ s.t. $g^n = 1$, if one exists, or ∞ otherwise.

A **homomorphism** (§1.6, p36) $\phi : G \to H$ is s.t. $\phi(xy) = \phi(x)\phi(y)$. An **isomorphism** is a bijective homomorphism.

The **kernel** (§1.6, ex. 14, p40) of a homomorphism is the inverse image of (**fiber over**) the identity element.

A nonempty subset H of G is a **subgroup** (§2.1, p46) ($H \leq G$) if it is closed under the group operation and inverses.

The subgroup of G **generated by** $A \subseteq G$ (§2.4, p62) is $\langle A\rangle = \bigcap_{A\subseteq H, H\leq G} H$.

2 Special Subgroups

The **centralizer** (§2.2, p49) of a nonempty subset A of G is $C_G(A) = \left\{g \in G \middle| \forall_{a\in A}.gag^{-1} = a\right\}$.

The **center** (§2.2, p50) of G is the centralizer of $A = G$.

The **normalizer** (§2.2, p50) of a nonempty subset A of G is $N_G(A) = \left\{g \in G \middle| \forall_{a\in A}.gag^{-1} \in A\right\}$.

A subgroup N of G is **normal** (§3.1, p82) ($N \trianglelefteq G$) if $N_G(N) = G$. (Equivalently, $\forall_{g\in G}.gN = Ng$; see Thm 6, p82.)

The a subgroup is normal iff it is kernel of a homomorphism. (§3.1, Prop 7, p82).

3 Group Actions

A **group action** (§1.7,p42) $- \cdot - : G \times A \to A$ (with A a *set*) obeys $\forall_{g,h\in G,a\in A}.g \cdot (h \cdot a) = (gh) \cdot a$ and $\forall_{a\in A}.1 \cdot a = a$.

Define $\sigma_g(-) = g \cdot -$ (p42). For each $g \in G$, σ_g is a permutation of A and $\phi = \{g \mapsto \sigma_g\}$ is a homomorphism $G \to S_A$.

The **kernel** (§1.7,p43) of an action is the set of left identities of $\cdot$.

The **stabilizer** (§2.2, p51) of $a \in A$ (A set) in group G is $G_a = \{g \in G | g \cdot a = a\}$.

4 Special Groups

The **dihedral group** (§1.2, p23) D_{2n} is the group formed by symmetries of a regular n-gon.

The **symmetric group** (§1.3, p29) S_Ω is the collection of all bijections $\Omega \to \Omega$ under composition. When $\Omega = \{1, \ldots, n\}$, S_Ω is denoted S_n.

The **quaternion group** (§1.5, p36) G_8 is $\{\pm1, \pm i, \pm j, \pm k\}$. $1a = a1 = a$, $(-1)a = a(-1) = -a$, $(-1)(-1) = 1$, $ii = jj = kk = -1$, $ij = k$, and $ji = -k$.

The **cyclic group** of order $n \in \mathbb{N}^+$ is $\mathbb{Z}_n \simeq \mathbb{Z}/n\mathbb{Z}$.

5 Quotient Groups

Given a homomorphism $\phi : G \to H$, the **quotient group** (§3.1, p76) $G/\ker\phi$ is the group of fibers of ϕ; if ϕ sends $X \mapsto a$ and $Y \mapsto b$ then $XY \mapsto ab$.

If $N \trianglelefteq G$ then $\pi : G \to G/N = \{g \mapsto gN\}$ is the **natural projection** of G onto G/N (§3.1, p83).

6 Isomorphism Theorems

7 Glossary

A **cycle** (p29), denoted $(a_1a_2 \ldots a_m)$, $(\forall_i.a_i \in \Omega)$ is an element of S_Ω which sends a_i to a_{i+1} and a_m to a_1.

A group is **cyclic** (§2.3, p54) if it is generated by a single element.

Given $H \leq G$ and $g \in G$, the set $\{gh|h \in H\}$ is a **left coset** of H, and $\{hg\}$ a **right coset** (§3.1, p77).

A nontrivial group is **simple** (§3.2, p102) if the only normal subgroups are trivial.

A group G is **solvable** (§3.4, p105) if $\exists_{\{G_i\}}$ s.t. $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ and each G_{i+1}/G_i is abelian.